SEQUENTIALLY COHEN-MACAULAY REES ALGEBRAS

NAOKI TANIGUCHI, TRAN THI PHUONG, NGUYEN THI DUNG, AND TRAN NGUYEN AN

Abstract. This paper studies the question of when the Rees algebras associated to arbitrary filtration of ideals are sequentially Cohen-Macaulay. Although this problem has been already investigated by [CGT], their situation is quite a bit of restricted, so we are eager to try the generalization of their results.

Contents

1. Introduction

The notion of sequentially Cohen-Macaulay property was originally introduced by R. P. Stanley ([St]) for Stanley-Reisner algebras and then it has been furiously explored by many researchers, say D. T. Cuong, N. T. Cuong, S. Goto, P. Schenzel and others (see [CC, CGT, GHS, Sch]), from the view point of not only combinatorics, but also commutative algebra. The purpose of this paper is to investigate the question of when the Rees algebras are sequentially Cohen-Macaulay, which has a previous research by [CGT]. In [CGT] they gave a characterization of the sequentially Cohen-Macaulay Rees algebras of $\mathfrak{m}$-primary ideals ([CGT, Theorem 5.2, Theorem 5.3]). However their situation is not entirely satisfactory, so we are eager to analyze the case where the ideal is not necessarily $\mathfrak{m}$-primary. More generally we want to deal with the sequentially Cohen-Macaulayness of the Rees modules since the sequentially Cohen-Macaulay property is defined for any finite modules over a Noetherian ring. Thus the main problem of this paper is when the Rees modules associated to arbitrary filtration of modules are sequentially Cohen-Macaulay.

2010 *Mathematics Subject Classification.*13A30, 13D45, 13E05, 13H10.

Key words and phrases: Dimension filtration, Sequentially Cohen-Macaulay module, Rees module.

The first author was partially supported by Grant-in-Aid for JSPS Fellows 26-126 and by JSPS Research Fellow. The second author was partially supported by JSPS KAKENHI 26400054. The third and the fourth author were partially supported by a Grant of Vietnam Institute for Advanced Study in Mathematics (VIASM) and Vietnam National Foundation for Science and Technology Development (NAFOSTED). .

Let R be a commutative Noetherian ring, $M \neq (0)$ a finitely generated R-module with $d = \dim_R M < \infty$. Then we consider a filtration

$$\mathcal{D} : D_0 := (0) \subsetneq D_1 \subsetneq D_2 \subsetneq \ldots \subsetneq D_\ell = M$$

of R-submodules of M, which we call *the dimension filtration of* M, if D_{i-1} is the largest R-submodule of D_i with $\dim_R D_{i-1} < \dim_R D_i$ for $1 \leq i \leq \ell$, here $\dim_R(0) = -\infty$ for convention. We note here that our notion of dimension filtration is based on [GHS] and slightly different from that of the original one given by P. Schenzel ([Sch]), however let us adopt the above definition throughout this paper. Then we say that M is *a sequentially Cohen-Macaulay R-module*, if the quotient module $C_i = D_i/D_{i-1}$ of D_i is a Cohen-Macaulay R-module for every $1 \leq i \leq \ell$. In particular, a Noetherian ring R is called *a sequentially Cohen-Macaulay ring*, if $\dim R < \infty$ and R is a sequentially Cohen-Macaulay module over itself.

Let us now state our results, explaining how this paper is organized. In Section 2 we sum up the notions of the sequentially Cohen-Macaulay properties and filtrations of ideals and modules. In Section 3 we shall give the proofs of the main results of this paper, which are stated as follows.

Suppose that R is a local ring with maximal ideal $\mathfrak{m}$. Let $\mathcal{F} = \{F_n\}_{n\in\mathbb{Z}}$ be a filtration of ideals of R such that $F_1 \neq R$, $\mathcal{M} = \{M_n\}_{n\in\mathbb{Z}}$ a $\mathcal{F}$-filtration of R-submodules of M. Then we put

$$\mathcal{R} = \sum_{n\geq 0} F_n t^n \subseteq R[t], \quad \mathcal{R}' = \sum_{n\in\mathbb{Z}} F_n t^n \subseteq R[t,t^{-1}], \quad \mathcal{G} = \mathcal{R}'/t^{-1}\mathcal{R}'$$

and call them *the Rees algebra, the extended Rees algebra and the associated graded ring of* $\mathcal{F}$, respectively. Similarly we set

$$\mathcal{R}(\mathcal{M}) = \sum_{n\geq 0} t^n \otimes M_n \subseteq R[t] \otimes_R M, \quad \mathcal{R}'(\mathcal{M}) = \sum_{n\in\mathbb{Z}} t^n \otimes M_n \subseteq R[t,t^{-1}] \otimes_R M$$

and

$$\mathcal{G}(\mathcal{M}) = \mathcal{R}'(\mathcal{M})/t^{-1}\mathcal{R}'(\mathcal{M})$$

which we call *the Rees module, the extended Rees module and the associated graded module of* $\mathcal{M}$, respectively. Here t stands for an indeterminate over R. We also assume that $\mathcal{R}$ is a Noetherian ring and $\mathcal{R}(\mathcal{M})$ is a finitely generated $\mathcal{R}$-module. Set

$$\mathcal{D}_i = \{M_n \cap D_i\}_{n\in\mathbb{Z}}, \quad \mathcal{C}_i = \{[(M_n \cap D_i) + D_{i-1}]/D_{i-1}\}_{n\in\mathbb{Z}}.$$

for every $1 \leq i \leq \ell$. Then $\mathcal{D}_i$ (resp. $\mathcal{C}_i$) is a $\mathcal{F}$-filtration of R-submodules of D_i (resp. C_i). With this notation the main results of this paper are the following, which are the natural generalization of the results [CGT, Theorem 5.2, Theorem 5.3].

Theorem 1.1. *The following conditions are equivalent.*
(1) $\mathcal{R}'(\mathcal{M})$ *is a sequentially Cohen-Macaulay* $\mathcal{R}'$*-module.*
(2) $\mathcal{G}(\mathcal{M})$ *is a sequentially Cohen-Macaulay* $\mathcal{G}$*-module and* $\{\mathcal{G}(\mathcal{D}_i)\}_{0\leq i\leq \ell}$ *is the dimension filtration of* $\mathcal{G}(\mathcal{M})$*.*

When this is the case, M *is a sequentially Cohen-Macaulay R-module.*

Let $\mathfrak{M}$ be a unique graded maximal ideal of $\mathcal{R}$. We set

$$\mathrm{a}(N) = \max\{n \in \mathbb{Z} \mid [\mathrm{H}^t_{\mathfrak{M}}(N)]_n \neq (0)\}$$

for a finitely generated graded $\mathcal{R}$-module N of dimension t, and call it *the a-invariant of* N (see [GW, DEFINITION (3.1.4)]). Here $\{[\mathrm{H}^t_{\mathfrak{M}}(N)]_n\}_{n\in\mathbb{Z}}$ stands for the homogeneous components of the t-th graded local cohomology module $\mathrm{H}^t_{\mathfrak{M}}(N)$ of N with respect to $\mathfrak{M}$.

Theorem 1.2. *Suppose that M is a sequentially Cohen-Macaulay R-module and $F_1 \nsubseteq \mathfrak{p}$ for every $\mathfrak{p} \in \mathrm{Ass}_R M$. Then the following conditions are equivalent.*
(1) *$\mathcal{R}(\mathcal{M})$ is a sequentially Cohen-Macaulay $\mathcal{R}$-module.*
(2) *$\mathcal{G}(\mathcal{M})$ is a sequentially Cohen-Macaulay $\mathcal{G}$-module, $\{\mathcal{G}(\mathcal{D}_i)\}_{0 \le i \le \ell}$ is the dimension filtration of $\mathcal{G}(\mathcal{M})$ and $\mathrm{a}(\mathcal{G}(\mathcal{C}_i)) < 0$ for every $1 \le i \le \ell$.*

When this is the case, $\mathcal{R}'(\mathcal{M})$ is a sequentially Cohen-Macaulay $\mathcal{R}'$-module.

In Section 4 we focus our attention on the case of graded rings. In the last section we will explore the application of Theorem 4.5 to the Stanley-Reisner algebras of shellable complexes (Theorem 5.2).

2. Preliminaries

In this section we summarize some basic results on sequentially Cohen-Macaulay properties and filtration of ideals and modules, which we will use throughout this paper.

Let R be a Noetherian ring, $M \neq (0)$ a finitely generated R-module of dimension d. We put

$$\mathrm{Assh}_R M = \{\mathfrak{p} \in \mathrm{Supp}_R M \mid \dim R/\mathfrak{p} = d\}.$$

For each $n \in \mathbb{Z}$, there exists the largest R-submodule M_n of M with $\dim_R M_n \le n$. Let

$$\begin{aligned} \mathcal{S}(M) &= \{\dim_R N \mid N \text{ is an } R\text{-submodule of } M, N \neq (0)\} \\ &= \{\dim R/\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Ass}_R M\}. \end{aligned}$$

We set $\ell = \sharp\mathcal{S}(M)$ and write $\mathcal{S}(M) = \{d_1 < d_2 < \cdots < d_\ell = d\}$. Let $D_i = M_{d_i}$ for each $1 \le i \le \ell$. We then have a filtration

$$\mathcal{D} : D_0 := (0) \subsetneq D_1 \subsetneq D_2 \subsetneq \ldots \subsetneq D_\ell = M$$

of R-submodules of M, which we call *the dimension filtration of M*. We put $C_i = D_i/D_{i-1}$ for every $1 \le i \le \ell$.

Definition 2.1 ([Sch, St])**.** We say that M is *a sequentially Cohen-Macaulay R-module*, if C_i is Cohen-Macaulay for every $1 \le i \le \ell$. The ring R is called *a sequentially Cohen-Macaulay ring*, if $\dim R < \infty$ and R is a sequentially Cohen-Macaulay module over itself.

The typical examples of sequentially Cohen-Macaulay ring is the Stanley-Reisner algebra $k[\Delta]$ of a shellable complex Δ over a field k. Also every one-dimensional Noetherian local ring is sequentially Cohen-Macaulay. Moreover, if M is a Cohen-Macaulay module over a Noetherian local ring, then M is sequentially Cohen-Macaulay, and the converse holds if M is unmixed.

Firstly let us note the non-zerodivisor characterization of sequentially Cohen-Macaulay modules.

Proposition 2.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, $M \neq (0)$ a finitely generated R-module. Let $x \in \mathfrak{m}$ be a non-zerodivisor on M. Then the following conditions are equivalent.*
(1) *M is a sequentially Cohen-Macaulay R-module.*
(2) *M/xM is a sequentially Cohen-Macaulay $R/(x)$-module and $\{D_i/xD_i\}_{0 \le i \le \ell}$ is the dimension filtration of M/xM.*

Proof. Notice that $x \in \mathfrak{m}$ is a non-zerodivisor on C_i and D_i for all $1 \le i \le \ell$ (See [Sch, Corollary 2.3]). Therefore we get a filtration

$$D_0/xD_0 = (0) \subsetneq D_1/xD_1 \subsetneq \cdots \subsetneq D_\ell/xD_\ell = M/xM$$

of $R/(x)$-submodules of M/xM. Then the assertion is a direct consequence of [GHS, Theorem 2.3]. $\square$

The implication (2) $\Rightarrow$ (1) is not true without the condition that $\{D_i/xD_i\}_{0 \le i \le \ell}$ is the dimension filtration of M/xM. For instance, let R be a 2-dimensional Noetherian local domain of depth 1 (e.g., Nagata's bad example [N]). Then $R/(x)$ is sequentially Cohen-Macaulay for every $0 \ne x \in R$, but R is not. Besides this, let I be an $\mathfrak{m}$-primary ideal in a regular local ring $(R, \mathfrak{m})$ of dimension 2. Then I is not a sequentially Cohen-Macaulay R-module, even though I/xI is, where $0 \ne x \in \mathfrak{m}$. These examples show that [Sch, Theorem 4.7] is not true in general.

From now on, we shall quickly review some preliminaries on filtrations of ideals and modules. Let R be a commutative ring, $\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}$ a filtration of ideals of R, that is, F_n is an ideal of R, $F_n \supseteq F_{n+1}$, $F_m F_n \subseteq F_{m+n}$ for all $m, n \in \mathbb{Z}$ and $F_0 = R$. Then we put

$$\mathcal{R} = \mathcal{R}(\mathcal{F}) = \sum_{n \ge 0} F_n t^n \subseteq R[t], \quad \mathcal{R}' = \mathcal{R}'(\mathcal{F}) = \sum_{n \in \mathbb{Z}} F_n t^n \subseteq R[t, t^{-1}]$$

and call them *the Rees algebra, the extended Rees algebra of R with respect to $\mathcal{F}$*, respectively. Here t stands for an indeterminate over R.

Let M be an R-module, $\mathcal{M} = \{M_n\}_{n \in \mathbb{Z}}$ an $\mathcal{F}$-filtration of R-submodules of M, that is, M_n is an R-submodule of M, $M_n \supseteq M_{n+1}$, $F_m M_n \subseteq M_{m+n}$ for all $m, n \in \mathbb{Z}$ and $M_0 = M$. We set

$$\mathcal{R}(\mathcal{M}) = \sum_{n \ge 0} t^n \otimes M_n \subseteq R[t] \otimes_R M, \quad \mathcal{R}'(\mathcal{M}) = \sum_{n \in \mathbb{Z}} t^n \otimes M_n \subseteq R[t, t^{-1}] \otimes_R M$$

which we call *the Rees module, the extended Rees module of M with respect to $\mathcal{M}$*, respectively, where

$$t^n \otimes M_n = \{t^n \otimes x \mid x \in M_n\} \subseteq R[t, t^{-1}] \otimes_R M$$

for all $n \in \mathbb{Z}$. Then $\mathcal{R}(\mathcal{M})$ (resp. $\mathcal{R}'(\mathcal{M})$) is a graded module over $\mathcal{R}$ (resp. $\mathcal{R}'$).

If $F_1 \ne R$, then we define *the associated graded ring $\mathcal{G}$ of R with respect to $\mathcal{F}$ and the associated graded module $\mathcal{G}(\mathcal{M})$ of M with respect to $\mathcal{M}$* as follows.

$$\mathcal{G} = \mathcal{G}(\mathcal{F}) = \mathcal{R}'/u\mathcal{R}', \quad \mathcal{G}(\mathcal{M}) = \mathcal{R}'(\mathcal{M})/u\mathcal{R}'(\mathcal{M}),$$

where $u = t^{-1}$. Then $\mathcal{G}(\mathcal{M})$ is a graded module over $\mathcal{G}$ and the composite map

$$\psi : \mathcal{R}(\mathcal{M}) \xrightarrow{i} \mathcal{R}'(\mathcal{M}) \xrightarrow{\varepsilon} \mathcal{G}(\mathcal{M})$$

is surjective and $\mathrm{Ker}\psi = u\mathcal{R}'(\mathcal{M}) \cap \mathcal{R}(\mathcal{M}) = u[\mathcal{R}(\mathcal{M})]_+$, where $[\mathcal{R}(\mathcal{M})]_+ = \sum_{n>0} t^n \otimes M_n$.

For the rest of this section, we assume that $F_1 \ne R$, $\mathcal{R} = \mathcal{R}(\mathcal{F})$ is Noetherian and $\mathcal{R}(\mathcal{M})$ is finitely generated. Then we have the following. The proof of Proposition 2.3 is based on the results [CGT, Proposition 5.1]. Since it plays an important role in this paper, let us give a brief proof for the sake of completeness.

Proposition 2.3. *The following assertions hold true.*

(1) *Let* $P \in \mathrm{Ass}_{\mathcal{R}}\mathcal{R}(\mathcal{M})$*. Then* $\mathfrak{p} \in \mathrm{Ass}_R M$*,* $P = \mathfrak{p}R[t] \cap \mathcal{R}$ *and*

$$\dim \mathcal{R}/P = \begin{cases} \dim R/\mathfrak{p} + 1 & \text{if } \dim R/\mathfrak{p} < \infty, F_1 \nsubseteq \mathfrak{p}, \\ \dim R/\mathfrak{p} & \text{otherwise}, \end{cases}$$

where $\mathfrak{p} = P \cap R$*.*

(2) $\mathfrak{p}R[t] \cap \mathcal{R} \in \mathrm{Ass}_{\mathcal{R}}\mathcal{R}(\mathcal{M})$ *for every* $\mathfrak{p} \in \mathrm{Ass}_R M$*.*

(3) *Suppose that* $M \neq (0)$*,* $d = \dim_R M < \infty$ *and there exists* $\mathfrak{p} \in \mathrm{Assh}_R M$ *such that* $F_1 \nsubseteq \mathfrak{p}$*. Then* $\dim_{\mathcal{R}} \mathcal{R}(\mathcal{M}) = d + 1$*.*

Proof. (1) Let $P \in \mathrm{Ass}_{\mathcal{R}}\mathcal{R}(\mathcal{M})$. Then $P \in \mathrm{Ass}_{\mathcal{R}} R[t] \otimes_R M$, so that $P = Q \cap \mathcal{R}$ for some

$$Q \in \mathrm{Ass}_{R[t]} R[t] \otimes_R M = \bigcup_{\mathfrak{p} \in \mathrm{Ass}_R M} \mathrm{Ass}_{R[t]} R[t]/\mathfrak{p}R[t].$$

Thus there exists $\mathfrak{p} \in \mathrm{Ass}_R M$ such that $\mathfrak{p} = Q \cap R$ and $Q = \mathfrak{p}R[t]$. Therefore $P = \mathfrak{p}R[t] \cap \mathcal{R}$, $\mathfrak{p} = P \cap R$. Put $\overline{R} = R/\mathfrak{p}$. Then $\overline{\mathcal{F}} = \{F_n\overline{R}\}_{n\in\mathbb{Z}}$ is a filtration of ideals of $\overline{R}$ and $\mathcal{R}/P \cong \mathcal{R}(\overline{\mathcal{F}})$ as graded R-algebras. Hence the assertion holds by [GN, Part II, Lemma (2.2)].

(2) Let $\mathfrak{p} \in \mathrm{Ass}_R M$. We write $\mathfrak{p} = (0) :_R x$ for some $x \in M$. Then $(0) :_{\mathcal{R}} \xi = \mathfrak{p}R[t] \cap \mathcal{R}$ where $\xi = 1 \otimes x \in [\mathcal{R}(\mathcal{M})]_0$.

(3) Follows from the assertions (1), (2). $\square$

Corollary 2.4. *Suppose that* R *is a local ring and* $M \neq (0)$*. Then*

$$\dim_{\mathcal{R}} \mathcal{R}(\mathcal{M}) = \begin{cases} \dim_R M + 1 & \text{if there exists } \mathfrak{p} \in \mathrm{Assh}_R M \text{ such that } F_1 \nsubseteq \mathfrak{p}, \\ \dim_R M & \text{otherwise}. \end{cases}$$

Similarly we are able to determine the structure of associated prime ideals of the extended Rees modules.

Proposition 2.5. *The following assertions hold true.*

(1) *Let* $P \in \mathrm{Ass}_{\mathcal{R}'}\mathcal{R}'(\mathcal{M})$*. Then* $\mathfrak{p} \in \mathrm{Ass}_R M$*,* $P = \mathfrak{p}R[t, t^{-1}] \cap \mathcal{R}'$ *and* $\dim \mathcal{R}/P = \dim R/\mathfrak{p} + 1$*, where* $\mathfrak{p} = P \cap R$*.*

(2) $\mathfrak{p}R[t, t^{-1}] \cap \mathcal{R}' \in \mathrm{Ass}_{\mathcal{R}'}\mathcal{R}'(\mathcal{M})$ *for every* $\mathfrak{p} \in \mathrm{Ass}_R M$*.*

(3) *Suppose that* $M \neq (0)$*. Then* $\dim_{\mathcal{R}'} \mathcal{R}'(\mathcal{M}) = \dim_R M + 1$*.*

Apply Proposition 2.5, we get the following.

Corollary 2.6. *Suppose* R *is a local ring and* $M \neq (0)$*. Then* $\dim_{\mathcal{G}} \mathcal{G}(\mathcal{M}) = \dim_R M$*.*

3. Proof of Theorem 1.1 and Theorem 1.2

This section aims to prove Theorem 1.1 and Theorem 1.2. In what follows, let $(R, \mathfrak{m})$ be a Noetherian local ring, $M \neq (0)$ a finitely generated R-module of dimension d. Let $\mathcal{F} = \{F_n\}_{n\in\mathbb{Z}}$ be a filtration of ideals of R with $F_1 \neq R$, $\mathcal{M} = \{M_n\}_{n\in\mathbb{Z}}$ a $\mathcal{F}$-filtration of R-submodules of M. We put $\mathfrak{a} = \mathcal{R}(\mathcal{F})_+ = \sum_{n>0} F_n t^n$.

Throughout this section we assume that $\mathcal{R} = \mathcal{R}(\mathcal{F})$ is a Noetherian ring and $\mathcal{R}(\mathcal{M})$ is finitely generated. Let $1 \leq i \leq \ell$. We set

$$\mathcal{D}_i = \{M_n \cap D_i\}_{n\in\mathbb{Z}}, \quad \mathcal{C}_i = \{[(M_n \cap D_i) + D_{i-1}]/D_{i-1}\}_{n\in\mathbb{Z}}.$$

Then $\mathcal{D}_i$ (resp. $\mathcal{C}_i$) is a $\mathcal{F}$-filtration of R-submodules of D_i (resp. C_i). Look at the following exact sequence

$$0 \to [\mathcal{D}_{i-1}]_n \to [\mathcal{D}_i]_n \to [\mathcal{C}_i]_n \to 0$$

of R-modules for all $n \in \mathbb{Z}$. We then have the exact sequences

$$0 \to \mathcal{R}(\mathcal{D}_{i-1}) \to \mathcal{R}(\mathcal{D}_i) \to \mathcal{R}(\mathcal{C}_i) \to 0$$
$$0 \to \mathcal{R}'(\mathcal{D}_{i-1}) \to \mathcal{R}'(\mathcal{D}_i) \to \mathcal{R}'(\mathcal{C}_i) \to 0 \text{ and}$$
$$0 \to \mathcal{G}(\mathcal{D}_{i-1}) \to \mathcal{G}(\mathcal{D}_i) \to \mathcal{G}(\mathcal{C}_i) \to 0$$

of graded modules. Since $\mathcal{R}(\mathcal{D}_i)$ is a finitely generated $\mathcal{R}$-module, so is $\mathcal{R}(\mathcal{C}_i)$.

Lemma 3.1. (*cf.* [CGT, Proposition 5.1]) *$\{\mathcal{R}'(\mathcal{D}_i)\}_{0 \le i \le \ell}$ is the dimension filtration of $\mathcal{R}'(\mathcal{M})$. If $F_1 \nsubseteq \mathfrak{p}$ for every $\mathfrak{p} \in \mathrm{Ass}_R M$, then $\{\mathcal{R}(\mathcal{D}_i)\}_{0 \le i \le \ell}$ is the dimension filtration of $\mathcal{R}(\mathcal{M})$.*

Proof. Let $1 \le i \le \ell$. Then $\dim_{\mathcal{R}'} \mathcal{R}'(\mathcal{D}_i) = d_i + 1$, since $D_i \ne (0)$. Let $P \in \mathrm{Ass}_{\mathcal{R}'} \mathcal{R}'(\mathcal{C}_i)$. Thanks to Proposition 2.5, we then have $\dim \mathcal{R}'/P = d_i + 1 = \dim_{\mathcal{R}'} \mathcal{R}'(\mathcal{C}_i)$. By using [GHS, Theorem 2.3], $\{\mathcal{R}'(\mathcal{D}_i)\}_{0 \le i \le \ell}$ is the dimension filtration of $\mathcal{R}'(\mathcal{M})$. Similarly we obtain the last assertion. $\square$

We now ready to prove Theorem 1.1.

Proof of Theorem 1.1. The equivalence of conditions (1) and (2) is similar to the proof of Proposition 2.2. Let us make sure of the last assertion. Look at the following exact sequences

$$0 \to \mathcal{R}'(\mathcal{C}_i) \xrightarrow{\varphi} R[t, t^{-1}] \otimes_R C_i \to X = \mathrm{Coker}\varphi \to 0$$

of graded $\mathcal{R}'$-modules for $1 \le i \le \ell$. Since $\mathcal{R}'(\mathcal{C}_i)$ is a Cohen-Macaulay $\mathcal{R}'$-module and the localization of X at $u = t^{-1}$ vanishes, we have $R[t, t^{-1}] \otimes_R C_i$ is Cohen-Macaulay. Therefore M is a sequentially Cohen-Macaulay R-module, because C_i is Cohen-Macaulay. $\square$

From now on, we focus our attention on the proof of Theorem 1.2. To do this, we need some auxiliaries.

Lemma 3.2. *Let $P \in \mathrm{Spec}\mathcal{R}$ such that $P \nsupseteq \mathfrak{a}$. If $\mathcal{G}(\mathcal{M})_P \ne (0)$ (resp. $\mathcal{R}(\mathcal{M})_P \ne (0)$ and $P \supseteq u\mathfrak{a}$), then $\mathcal{R}(\mathcal{M})_P \ne (0)$ (resp. $\mathcal{G}(\mathcal{M})_P \ne (0)$). When this is the case, the following assertions hold true.*

(1) *$\mathcal{R}(\mathcal{M})_P$ is a Cohen-Macaulay $\mathcal{R}_P$-module if and only if $\mathcal{G}(\mathcal{M})_P$ is a Cohen-Macaulay $\mathcal{G}_P$-module.*
(2) $\dim_{\mathcal{R}_P} \mathcal{R}(\mathcal{M})_P = \dim_{\mathcal{R}_P} \mathcal{G}(\mathcal{R})_P + 1$.

Proof. Let $P \in \mathrm{Spec}\mathcal{R}$ such that $P \nsupseteq \mathfrak{a}$, but $P \supseteq u\mathfrak{a}$. We choose a homogeneous element $\xi = at^n \in \mathfrak{a} \setminus P$ where $n > 0$, $a \in F_n$. Then we get $x = u\xi = at^{n-1} \in P$, since $P \supseteq u\mathfrak{a}$.

Claim 3.3. *If $Q \in \mathrm{Ass}_{\mathcal{R}} \mathcal{R}(\mathcal{M})$ such that $Q \subseteq P$, then $x \notin Q$. Therefore x is a non-zerodivisor on $\mathcal{R}(\mathcal{M})_P$.*

Proof of Claim 3.3. We assume that there exists $Q \in \mathrm{Ass}_{\mathcal{R}} \mathcal{R}(\mathcal{M})$ such that $Q \subseteq P$, but $x \in Q$. Write $Q = (0) :_{\mathcal{R}} \eta$ where $\eta = t^\ell \otimes m$ ($\ell \in \mathbb{Z}$, $m \in M_\ell$). Then we have $\xi = at^n \in (0) :_{\mathcal{R}} \eta = Q \subseteq P$, which implies a contradiction. $\square$

Since $P \not\supseteq \mathfrak{a}$, we get $\mathcal{R}_P = \mathcal{R}'_P$ and $\mathcal{R}(\mathcal{M})_P = \mathcal{R}'(\mathcal{M})_P$. Therefore

$$(u\mathfrak{a})\mathcal{R}_P = (u\mathfrak{a})\mathcal{R}'_P = u\mathcal{R}'_P = x\mathcal{R}'_P \text{ and } (u\mathfrak{a})\mathcal{R}(\mathcal{M}) \subseteq u[\mathcal{R}(\mathcal{M})]_+.$$

Hence $[u\mathcal{R}(\mathcal{M})_+]_P = x\mathcal{R}'(\mathcal{M})_P = x\mathcal{R}(\mathcal{M})_P$, so that

$$\mathcal{R}(\mathcal{M})_P / x\mathcal{R}(\mathcal{M})_P \cong \mathcal{G}(\mathcal{M})_P$$

as $\mathcal{R}_P$-modules. On the other hand, let $P \in \operatorname{Spec}\mathcal{R}$ such that $\mathcal{G}(\mathcal{M})_P \neq (0)$. Then $P \supseteq u\mathfrak{a}$, since $u\mathfrak{a} = u\mathcal{R}' \cap \mathcal{R} = \operatorname{Ker}(\mathcal{R} \xrightarrow{i} \mathcal{R}' \xrightarrow{\varepsilon} \mathcal{G})$. Therefore the assertions immediately come from the above isomorphism. $\square$

Here we need the following fact, which was originally given by G. Faltings.

Fact 3.4 ([F])**.** Let I be an ideal of R and $t \in \mathbb{Z}$. Consider the following two conditions.
(1) There exists an integer $\ell > 0$ such that $I^\ell {\cdot} \mathrm{H}^i_\mathfrak{m}(M) = (0)$ for each $i \neq t$.
(2) $M_\mathfrak{p}$ is a Cohen Macaulay $R_\mathfrak{p}$-module and $t = \dim_{R_\mathfrak{p}} M_\mathfrak{p} + \dim R/\mathfrak{p}$ for every $\mathfrak{p} \in \operatorname{Supp}_R M$ but $\mathfrak{p} \not\supseteq I$.

Then the implication (1) $\Rightarrow$ (2) holds true. The converse holds, if R is a homomorphic image of a Gorenstein local ring.

For an arbitrary ideal I of a graded ring, we define I^* to be the ideal generated by every homogeneous element in I. Let $\mathfrak{M}$ be a unique graded maximal ideal of $\mathcal{R}$.

Although a part of the proof of Proposition 3.5 is due to the result [TI], we note the brief proof for the sake of completeness.

Proposition 3.5. *Suppose that $\mathrm{H}^i_\mathfrak{M}(\mathcal{G}(\mathcal{M}))$ is a finitely graded $\mathcal{R}$-module for all $i \neq d$. Then $\mathrm{H}^i_\mathfrak{M}(\mathcal{R}(\mathcal{M}))$ is a finitely graded $\mathcal{R}$-module for all $i \neq d+1$.*

Proof. Passing to the completion we may assume that R is a homomorphic image of a Gorenstein local ring, and so is $\mathcal{R}_\mathfrak{M}$. Thanks to the local duality theorem, it is enough to show that there exists an integer $\ell > 0$ such that $\mathfrak{a}^\ell {\cdot} \mathrm{H}^i_\mathfrak{M}(\mathcal{R}(M)) = (0)$ for every $i \neq d+1$. To see this, let $P \in \operatorname{Supp}_\mathcal{R}\mathcal{R}(M)$ such that $P \not\supseteq \mathfrak{a}$ and $P \subseteq \mathfrak{M}$. Put $L = u\mathfrak{a} = u\mathcal{R}' \cap \mathcal{R}$.

Claim 3.6. $\sqrt{P^* + L} \not\supseteq \mathfrak{a}$.

Proof of Claim 3.6. Suppose that $P^* + L \supseteq \mathfrak{a}^\ell$ for some $\ell > 0$. Since $\mathcal{R}/\mathfrak{a}^\ell$ is finitely graded, we can choose an integer $s > 0$ such that $[\mathcal{R}/\mathfrak{a}^\ell]_n = (0)$ for all $n \geq s$. Then

$$\mathcal{R}_n = F_n t^n \subseteq [P^*]_n + F_{n+1} t^n$$

for all $n \geq s$. On the other hand, for each $n \geq 0$, we set

$$I_n = \{a \in R \mid at^n \in P^*\}.$$

Then I_n is an ideal of R and $I_n \subseteq F_n$ and $I_n \supseteq I_{n+1}$ for all $n \geq 0$. Hence $F_n \subseteq I_n + F_k$ for all $n \geq s$, $k \in \mathbb{Z}$. Since $\mathcal{R}$ is Noetherian, we get $\mathcal{R}^{(d)} = R[F_d t^d]$ for some $d > 0$, so that $F_{d\ell} = (F_d)^\ell$ for all $\ell > 0$. We then have

$$F_n \subseteq \bigcap_{\ell > 0} [I_n + (F_d)^\ell] = I_n$$

for all $n \geq s$, whence $\mathcal{R}_n \subseteq P^*$. Thus

$$\mathfrak{a}^s \subseteq \sum_{n \geq s} \mathcal{R}_n \subseteq P^* \subseteq P.$$

which is impossible, because $\mathfrak{a} \nsubseteq P$. $\square$

Therefore we can take $Q \in \mathrm{Min}_{\mathcal{R}} \mathcal{R}/[P^* + L]$ such that $\mathfrak{a} \nsubseteq Q \subseteq \mathfrak{M}$. Then $\mathcal{R}(\mathcal{M})_Q \neq (0)$, because $\mathcal{R}(\mathcal{M})_{P^*} \neq (0)$ and $P^* \subseteq Q$. Thanks to Lemma 3.2, $\mathcal{G}(\mathcal{M})_Q \neq (0)$. Then $\mathcal{G}(\mathcal{M})_Q$ is Cohen-Macaulay and $\dim_{\mathcal{R}_Q} \mathcal{G}(\mathcal{M})_Q + \dim \mathcal{R}_{\mathfrak{M}}/Q\mathcal{R}_{\mathfrak{M}} = d$ by using Fact 3.4. Hence $\mathcal{R}(\mathcal{M})_Q$ is Cohen-Macaulay and $\dim_{\mathcal{R}_Q} \mathcal{R}(\mathcal{M})_Q + \dim \mathcal{R}_{\mathfrak{M}}/Q\mathcal{R}_{\mathfrak{M}} = d+1$ by Lemma 3.2.

Since $P^* \subseteq Q$, $\mathcal{R}(M)_{P^*}$ is Cohen-Macaulay, so is $\mathcal{R}(M)_P$. We also have

$$\begin{aligned} d+1 &= \dim_{\mathcal{R}_Q} \mathcal{R}(M)_Q + \dim \mathcal{R}_{\mathfrak{M}}/Q\mathcal{R}_{\mathfrak{M}} \\ &= (\dim_{\mathcal{R}_{P^*}} \mathcal{R}(M)_{P^*} + \dim \mathcal{R}_Q/P^*\mathcal{R}_Q) + (\dim \mathcal{R}_{\mathfrak{M}}/P^*\mathcal{R}_{\mathfrak{M}} - \dim \mathcal{R}_Q/P^*\mathcal{R}_Q) \\ &= \dim_{\mathcal{R}_{P^*}} \mathcal{R}(M)_{P^*} + \dim \mathcal{R}_{\mathfrak{M}}/P^*\mathcal{R}_{\mathfrak{M}} \\ &= \dim_{\mathcal{R}_P} \mathcal{R}(M)_P + \dim \mathcal{R}_{\mathfrak{M}}/P\mathcal{R}_{\mathfrak{M}}. \end{aligned}$$

Thanks to Fact 3.4 again, there exists $\ell > 0$ such that

$$\mathfrak{a}^{\ell} \cdot \mathrm{H}^i_{\mathfrak{M}}(\mathcal{R}(\mathcal{M})) = (0) \quad \text{for each} \quad i \neq d+1$$

which shows $\mathrm{H}^i_{\mathfrak{M}}(\mathcal{R}(\mathcal{M}))$ is finitely graded. $\square$

We set

$$\mathrm{a}(N) = \max\{n \in \mathbb{Z} \mid [\mathrm{H}^t_{\mathfrak{M}}(N)]_n \neq (0)\}$$

for a finitely generated graded $\mathcal{R}$-module N of dimension t, and call it *the a-invariant of* N (see [GW, DEFINITION (3.1.4)]). With this notation we have the following.

Lemma 3.7. *The following assertions hold true.*
(1) $[\mathrm{H}^{d+1}_{\mathfrak{M}}(\mathcal{R}(\mathcal{M}))]_n = (0)$ *for all* $n \geq 0$.
(2) *If* $[\mathrm{H}^{d+1}_{\mathfrak{M}}(\mathcal{R}(\mathcal{M}))]_{-1} = (0)$, *then* $\mathrm{H}^{d+1}_{\mathfrak{M}}(\mathcal{R}(\mathcal{M})) = (0)$.
Consequently $\mathrm{a}(\mathcal{R}(\mathcal{M})) = -1$, *if* $\dim_{\mathcal{R}} \mathcal{R}(\mathcal{M}) = d+1$.

Proof. We look at the following exact sequences

$$0 \to L \to \mathcal{R}(\mathcal{M}) \to M \to 0$$

$$0 \to L(1) \to \mathcal{R}(\mathcal{M}) \to \mathcal{G}(\mathcal{M}) \to 0$$

of graded $\mathcal{R}$-modules, where $L = \mathcal{R}(\mathcal{M})_+$. By applying the local cohomology functors to the above sequences, we get

$$\mathrm{H}^d_{\mathfrak{m}}(M) \to \mathrm{H}^{d+1}_{\mathfrak{M}}(L) \to \mathrm{H}^{d+1}_{\mathfrak{M}}(\mathcal{R}(\mathcal{M})) \to 0$$

and

$$\mathrm{H}^d_{\mathfrak{M}}(\mathcal{G}(\mathcal{M})) \to \mathrm{H}^{d+1}_{\mathfrak{M}}(L)(1) \to \mathrm{H}^{d+1}_{\mathfrak{M}}(\mathcal{R}(\mathcal{M})) \to 0.$$

Thus

$$[\mathrm{H}^{d+1}_{\mathfrak{M}}(L)]_n \cong [\mathrm{H}^{d+1}_{\mathfrak{M}}(\mathcal{R}(\mathcal{M}))]_n \quad \text{for} \quad n \neq 0, \text{ and}$$

$$[\mathrm{H}^{d+1}_{\mathfrak{M}}(L)]_{n+1} \to [\mathrm{H}^{d+1}_{\mathfrak{M}}(\mathcal{R}(\mathcal{M}))]_n \to 0 \quad \text{for} \quad n \in \mathbb{Z}.$$

Therefore $[\mathrm{H}^{d+1}_{\mathfrak{M}}(\mathcal{R}(\mathcal{M}))]_n = (0)$ for $n \geq 0$, because $\mathrm{H}^{d+1}_{\mathfrak{M}}(\mathcal{R}(\mathcal{M}))$ is Artinian. Moreover we have

$$[H^{d+1}_{\mathfrak{M}}(\mathcal{R}(\mathcal{M}))]_{-1} \to [\mathrm{H}^{d+1}_{\mathfrak{M}}(\mathcal{R}(\mathcal{M}))]_n \to 0$$

for $n < 0$, so we get the assertion (2). $\square$

We finally arrive at the following Theorem 3.8 which is a module version of the results [GN, Part II, Theorem (1.1)], [V, Theorem 1.1] (see also [TI, Theorem 1.1], [GS, Theorem (1.1)]).

Theorem 3.8. *The following conditions are equivalent.*
(1) $\mathcal{R}(\mathcal{M})$ *is a Cohen-Macaulay* $\mathcal{R}$*-module and* $\dim_{\mathcal{R}} \mathcal{R}(\mathcal{M}) = d+1$.
(2) $\mathrm{H}^i_{\mathfrak{M}}(\mathcal{G}(\mathcal{M})) = [\mathrm{H}^i_{\mathfrak{M}}(\mathcal{G}(\mathcal{M}))]_{-1}$ *for every* $i < d$ *and* $\mathrm{a}(\mathcal{G}(\mathcal{M})) < 0$.
When this is the case, $[\mathrm{H}^i_{\mathfrak{M}}(\mathcal{G}(\mathcal{M}))]_{-1} \cong \mathrm{H}^i_{\mathfrak{m}}(M)$ *as* R*-modules for all* $i < d$.

Proof. Consider the following exact sequences

$$(*)\ \cdots \to \mathrm{H}^i_{\mathfrak{m}}(L) \to \mathrm{H}^i_{\mathfrak{M}}(\mathcal{R}(\mathcal{M})) \to \mathrm{H}^i_{\mathfrak{m}}(M) \to \mathrm{H}^{i+1}_{\mathfrak{M}}(L) \to \mathrm{H}^{i+1}_{\mathfrak{M}}(\mathcal{R}(\mathcal{M})) \to \cdots$$

$$(**)\ \cdots \to \mathrm{H}^i_{\mathfrak{m}}(L)(1) \to \mathrm{H}^i_{\mathfrak{M}}(\mathcal{R}(\mathcal{M})) \to \mathrm{H}^i_{\mathfrak{M}}(\mathcal{G}(\mathcal{M})) \to \mathrm{H}^{i+1}_{\mathfrak{M}}(L)(1) \to \mathrm{H}^{i+1}_{\mathfrak{M}}(\mathcal{R}(\mathcal{M})) \to \cdots$$

for each $i < d$.

Firstly we assume that $\mathcal{R}(\mathcal{M})$ is a Cohen-Macaulay $\mathcal{R}$-module of dimension $d+1$. Then

$$\mathrm{H}^i_{\mathfrak{m}}(M) \cong \mathrm{H}^{i+1}_{\mathfrak{M}}(L) \quad \text{and} \quad \mathrm{H}^i_{\mathfrak{M}}(\mathcal{G}(\mathcal{M})) \cong \mathrm{H}^{i+1}_{\mathfrak{M}}(L)(1)$$

for $i < d$. Therefore we get $\mathrm{H}^i_{\mathfrak{M}}(\mathcal{G}(\mathcal{M})) = [\mathrm{H}^i_{\mathfrak{M}}(\mathcal{G}(\mathcal{M}))]_{-1}$ and $[\mathrm{H}^i_{\mathfrak{M}}(\mathcal{G}(\mathcal{M}))]_{-1} \cong \mathrm{H}^i_{\mathfrak{m}}(M)$ as R-modules. Since $\mathcal{R}(\mathcal{M})$ is Cohen-Macaulay, we have

$$0 \to \mathrm{H}^d_{\mathfrak{m}}(M) \to \mathrm{H}^{d+1}_{\mathfrak{M}}(L) \to \mathrm{H}^{d+1}_{\mathfrak{M}}(\mathcal{R}(\mathcal{M})) \to 0$$

$$0 \to \mathrm{H}^d_{\mathfrak{M}}(\mathcal{G}(\mathcal{M})) \to \mathrm{H}^{d+1}_{\mathfrak{M}}(L)(1).$$

Therefore $\mathrm{a}(\mathcal{G}(\mathcal{M})) < 0$ by using Lemma 3.7.

Conversely, let $i < d$. Thanks to the above sequences $(*)$, $(**)$ and our hypothesis, we get

$$[\mathrm{H}^{i+1}_{\mathfrak{M}}(L)]_{n+1} \cong [\mathrm{H}^{i+1}_{\mathfrak{M}}(\mathcal{R}(\mathcal{M}))]_n$$

$$[\mathrm{H}^{i+1}_{\mathfrak{M}}(L)]_{n+1} \cong [\mathrm{H}^{i+1}_{\mathfrak{M}}(\mathcal{R}(\mathcal{M}))]_{n+1}$$

for each $n \geq 0$. Hence $[\mathrm{H}^i_{\mathfrak{M}}(\mathcal{R}(\mathcal{M}))]_n = (0)$ for $n \geq 0$, since $\mathrm{H}^{i+1}_{\mathfrak{M}}(\mathcal{R}(\mathcal{M}))$ is Artinian. Moreover, we then have

$$0 \to [\mathrm{H}^{i+1}_{\mathfrak{M}}(\mathcal{R}(\mathcal{M}))]_n \to [\mathrm{H}^{i+1}_{\mathfrak{M}}(\mathcal{R}(\mathcal{M}))]_{n-1}.$$

for $n < 0$ by above sequences $(*)$ and $(**)$. Thanks to Proposition 3.5, $\mathrm{H}^{i+1}_{\mathfrak{M}}(\mathcal{R}(\mathcal{M}))$ is a finitely graded $\mathcal{R}$-module for $i < d$. Whence $[\mathrm{H}^{i+1}_{\mathfrak{M}}(\mathcal{R}(\mathcal{M}))]_n = (0)$, which shows $\mathrm{H}^{i+1}_{\mathfrak{M}}(\mathcal{R}(\mathcal{M})) = (0)$ for all $i < d$. Hence $\mathcal{R}(\mathcal{M})$ is a Cohen-Macaulay $\mathcal{R}$-module of dimension $d+1$. □

Corollary 3.9. *Suppose that* M *is a Cohen-Macaulay* R*-module. Then the following conditions are equivalent.*
(1) $\mathcal{R}(\mathcal{M})$ *is a Cohen-Macaulay* $\mathcal{R}$*-module and* $\dim_{\mathcal{R}} \mathcal{R}(\mathcal{M}) = d+1$.
(2) $\mathcal{G}(\mathcal{M})$ *is a Cohen-Macaulay* $\mathcal{G}$*-module and* $\mathrm{a}(\mathcal{G}(\mathcal{M})) < 0$.

We now reach the goal of this section.

Proof of Theorem 1.2. Thanks to Lemma 3.1, $\mathcal{R}(\mathcal{M})$ is a sequentially Cohen-Macaulay $\mathcal{R}$-module if and only if $\mathcal{R}(\mathcal{C}_i)$ is Cohen-Macaulay for every $1 \leq i \leq \ell$. The latter condition is equivalent to saying that $\mathcal{G}(\mathcal{C}_i)$ is a Cohen-Macaulay $\mathcal{G}$-module and $\mathrm{a}(\mathcal{G}(\mathcal{C}_i)) < 0$ for all $1 \leq i \leq \ell$ by Corollary 3.9. Hence we get the equivalence between (1) and (2). □

We close this section by stating the ring version of Theorem 1.1 and Theorem 1.2. Let $(R, \mathfrak{m})$ be a Noetherian local ring, $\mathcal{F} = \{F_n\}_{n\in\mathbb{Z}}$ a filtration of ideals of R such that $F_1 \neq R$. We assume that $\mathcal{R} = \mathcal{R}(\mathcal{F})$ is a Noetherian ring. Let $\{D_i\}_{0\le i\le \ell}$ be the dimension filtration of R. Then $\mathcal{D}_i = \{F_n \cap D_i\}_{n\in\mathbb{Z}}$ (resp. $\mathcal{C}_i = \{[F_n \cap D_i + D_{i-1}]/D_{i-1}\}_{n\in\mathbb{Z}}$) is a $\mathcal{F}$-filtration of D_i (resp. C_i) for all $1 \le i \le \ell$.

Theorem 3.10. *The following conditions are equivalent.*
(1) *$\mathcal{R}'$ is a sequentially Cohen-Macaulay ring.*
(2) *$\mathcal{G}$ is a sequentially Cohen-Macaulay ring and $\{\mathcal{G}(\mathcal{D}_i)\}_{0\le i\le \ell}$ is the dimension filtration of $\mathcal{G}$.*

When this is the case, R is a sequentially Cohen-Macaulay ring.

Theorem 3.11. *Suppose that R is a sequentially Cohen-Macaulay ring and $F_1 \nsubseteq \mathfrak{p}$ for every $\mathfrak{p} \in \mathrm{Ass} R$. Then the following conditions are equivalent.*
(1) *$\mathcal{R}$ is a sequentially Cohen-Macaulay ring.*
(2) *$\mathcal{G}$ is a sequentially Cohen-Macaulay ring, $\{\mathcal{G}(\mathcal{D}_i)\}_{0\le i\le \ell}$ is the dimension filtration of $\mathcal{G}$ and $\mathrm{a}(\mathcal{G}(\mathcal{C}_i)) < 0$ for all $1 \le i \le \ell$.*

When this is the case, $\mathcal{R}'$ is a sequentially Cohen-Macaulay ring.

4. Sequentially Cohen-Macaulay property in $E^\natural$

In this section let $R = \sum_{n\ge 0} R_n$ be a $\mathbb{Z}$-graded ring. We put $F_n = \sum_{k\ge n} R_k$ for all $n \in \mathbb{Z}$. Then F_n is a graded ideal of R, $\mathcal{F} = \{F_n\}_{n\in\mathbb{Z}}$ is a filtration of ideals of R and $F_1 := R_+ \neq R$. Let E be a graded R-module with $E_n = (0)$ for all $n < 0$. Put $E_{(n)} = \sum_{k\ge n} E_k$ for all $n \in \mathbb{Z}$. Then $E_{(n)}$ is a graded R-submodule of E, $\mathcal{E} = \{E_{(n)}\}_{n\in\mathbb{Z}}$ is an $\mathcal{F}$-filtration of R-submodules of E and $E_{(0)} = E$. Then we have $R = \mathcal{G}(\mathcal{F})$ and $E = \mathcal{G}(\mathcal{E})$. Set $R^\natural := \mathcal{R}(\mathcal{F})$ and $E^\natural := \mathcal{R}(\mathcal{E})$.

Suppose that R is a Noetherian ring and $E \neq (0)$ is a finitely generated graded R-module with $d = \dim_R E < \infty$. Notice that $R^\natural$ is Noetherian and $E^\natural$, $\mathcal{R}'(\mathcal{E})$ are finitely generated.

We note the following.

Lemma 4.1. *The following assertions hold true.*
(1) $\dim_{\mathcal{R}'} \mathcal{R}'(\mathcal{E}) = d + 1$.
(2) *Suppose that there exists $\mathfrak{p} \in \mathrm{Assh}_R E$ such that $F_1 \nsubseteq \mathfrak{p}$. Then $\dim_{R^\natural} E^\natural = d + 1$.*

Proof. See Proposition 2.3, Proposition 2.5. □

Let $D_0 \subsetneq D_1 \subsetneq \ldots \subsetneq D_\ell = E$ be the dimension filtration of E. We set $C_i = D_i/D_{i-1}$, $d_i = \dim_R D_i$ for every $1 \le i \le \ell$. Then D_i is a graded R-submodule of E for all $0 \le i \le \ell$. Let $1 \le i \le \ell$. Then from the exact sequence

$$0 \to [D_{i-1}]_{(n)} \to [D_i]_{(n)} \to [C_i]_{(n)} \to 0$$

of graded R-modules for all $n \in \mathbb{Z}$, we get the exact sequences

$$0 \to \mathcal{R}(\mathcal{D}_{i-1}) \to \mathcal{R}(\mathcal{D}_i) \to \mathcal{R}(\mathcal{C}_i) \to 0$$

$$0 \to \mathcal{R}'(\mathcal{D}_{i-1}) \to \mathcal{R}'(\mathcal{D}_i) \to \mathcal{R}'(\mathcal{C}_i) \to 0 \quad \text{and}$$

$$0 \to \mathcal{G}(\mathcal{D}_{i-1}) \to \mathcal{G}(\mathcal{D}_i) \to \mathcal{G}(\mathcal{C}_i) \to 0$$

of graded modules, where $\mathcal{D}_i = \{[D_i]_{(n)}\}_{n\in\mathbb{Z}}$, $\mathcal{C}_i = \{[C_i]_{(n)}\}_{n\in\mathbb{Z}}$. By the same technique as in the proof of Lemma 3.1, we obtain the dimension filtration of $\mathcal{R}'(E)$ and $E^\natural$ as follows.

Lemma 4.2. *$\{\mathcal{R}'(\mathcal{D}_i)\}_{0\le i\le \ell}$ is the dimension filtration of $\mathcal{R}'(\mathcal{E})$. If $F_1 \nsubseteq \mathfrak{p}$ for every $\mathfrak{p} \in \mathrm{Ass}_R E$, then $\{\mathcal{R}(\mathcal{D}_i)\}_{0\le i\le \ell}$ is the dimension filtration of $E^\natural$.*

Hence we get the following, which characterize the sequentially Cohen-Macaulayness of $\mathcal{R}'(\mathcal{E})$.

Proposition 4.3. *The following conditions are equivalent.*
(1) *$\mathcal{R}'(\mathcal{E})$ is a sequentially Cohen-Macaulay $\mathcal{R}'$-module.*
(2) *E is a sequentially Cohen-Macaulay R-module.*

Proof. (1) $\Rightarrow$ (2) Follows from the fact that $C_i = \mathcal{G}(\mathcal{C}_i)$ for each $1 \le i \le \ell$.

(2) $\Rightarrow$ (1) We get $\mathcal{G}(\mathcal{C}_i)$ is Cohen-Macaulay for all $1 \le i \le \ell$. Let $Q \in \mathrm{Supp}_{\mathcal{R}'}\mathcal{R}'(\mathcal{C}_i)$. We may assume $u \notin Q$. Then $\mathcal{R}'(\mathcal{C}_i)_u = R[t,t^{-1}] \otimes_R C_i$ is Cohen-Macaulay since C_i is Cohen-Macaulay. Hence $\mathcal{R}'(\mathcal{C}_i)_Q$ is a Cohen-Macaulay $\mathcal{R}'_Q$-module. $\square$

Now we study the question of when $E^\natural$ is sequentially Cohen-Macaulay. The key is the following.

Lemma 4.4. *Suppose R_0 is a local ring, E is a Cohen-Macaulay R-module and $F_1 \nsubseteq \mathfrak{p}$ for some $\mathfrak{p} \in \mathrm{Assh}_R E$. Then the following conditions are equivalent.*
(1) *$E^\natural$ is a Cohen-Macaulay $R^\natural$-module.*
(2) $\mathrm{a}(E) < 0$.

Proof. Let $P = \mathfrak{m}R + R_+$, where $\mathfrak{m}$ denotes the maximal ideal of R_0. Then $P \supseteq F_1$. Since $R_+(E_{(n)}/E_{(n+1)}) = (0)$, $R_+(F_n/F_{n+1}) = (0)$ for all $n \in \mathbb{Z}$, we have

$$E = \mathcal{G}(\mathcal{E}) \cong \mathcal{G}(\mathcal{E}_P), \quad R = \mathcal{G}(\mathcal{F}) \cong \mathcal{G}(\mathcal{F}_P).$$

Suppose that $E^\natural$ is a Cohen-Macaulay $R^\natural$-module. Then $\mathcal{R}(\mathcal{E}_P)$ is Cohen-Macaulay and $\dim_{\mathcal{R}(R_P)} \mathcal{R}(\mathcal{E}_P) = d+1$, whence $\mathcal{G}(\mathcal{E}_P)$ is Cohen-Macaulay and $\mathrm{a}(\mathcal{G}(\mathcal{E}_P)) < 0$. Therefore we get $\mathrm{a}(E) < 0$.

On the other hand, suppose that $\mathrm{a}(E) < 0$. Then $\mathcal{R}(\mathcal{E}_P)$ is a Cohen-Macaulay $\mathcal{R}(R_P)$-module of dimension $d+1$. Thus $\mathcal{R}(\mathcal{E})_P$ is Cohen-Macaulay. Now we regard $\mathcal{R}$ as a $\mathbb{Z}^2$-graded ring with the $\mathbb{Z}^2$-grading as follows:

$$\mathcal{R}_{(i,j)} = \begin{cases} R_i t^j & i \ge j \ge 0 \\ (0) & \textit{otherwise}. \end{cases}$$

Moreover we set

$$\mathcal{R}(\mathcal{E})_{(i,j)} = \begin{cases} t^j \otimes E_i & i \ge j \ge 0 \\ (0) & \textit{otherwise}, \end{cases}$$

where $t^j \otimes E_i = \{t^j \otimes x \mid x \in E_i\}$. Then $\mathcal{R}(\mathcal{E})$ is a $\mathbb{Z}^2$-graded $\mathcal{R}$-module with the above grading $\mathcal{R}(\mathcal{E})_{(i,j)}$. Notice that $\mathcal{R}_{(0,0)} = R_0$ is a local ring, so that $\mathcal{R}$ is H-local, that is $\mathbb{Z}^2$-graded ring $\mathcal{R}$ has a unique graded maximal ideal L. Then we get $P \subseteq L$, whence $L \cap R = P$. Therefore $\mathcal{R}(\mathcal{E})_L$ is a Cohen-Macaulay $\mathcal{R}_L$-module, so that $E^\natural$ is Cohen-Macaulay. $\square$

Our answer is the following.

Theorem 4.5. *Suppose that R_0 is a local ring, E is a sequentially Cohen-Macaulay R-module and $F_1 \not\subseteq \mathfrak{p}$ for every $\mathfrak{p} \in \mathrm{Ass}_R E$. Then the following conditions are equivalent.*

(1) *$E^\natural$ is a sequentially Cohen-Macaulay $R^\natural$-module.*
(2) *$\mathrm{a}(C_i) < 0$ for all $1 \le i \le \ell$.*

5. Application –Stanley-Reisner algebras–

In this section, let $V = \{1, 2, \ldots, n\}$ $(n > 0)$ be a vertex set, Δ a simplicial complex on V such that $\Delta \neq \emptyset$. We denote $\mathcal{F}(\Delta)$ a set of facets of Δ and $m = \sharp\mathcal{F}(\Delta)$ (> 0) its cardinality. Let $S = k[X_1, X_2, \ldots, X_n]$ be a polynomial ring over a field k, $R = k[\Delta] = S/I_\Delta$ the *Stanley-Reisner ring of* Δ of dimension d, where $I_\Delta = (X_{i_1}X_{i_2}\cdots X_{i_r} \mid \{i_1 < i_2 < \cdots < i_r\} \notin \Delta)$ is the *Stanley-Reisner ideal of* R.

We consider the Stanley-Reisner ring $R = \sum_{n \ge 0} R_n$ as a $\mathbb{Z}$-graded ring and put

$$I_n = \sum_{k \ge n} R_k = \mathfrak{m}^n \quad \text{for all} \quad n \in \mathbb{Z}$$

where $\mathfrak{m} = R_+ = \sum_{n>0} R_n$ is a graded maximal ideal of R. Then $\mathcal{I} = \{I_n\}_{n\in\mathbb{Z}}$ is a $\mathfrak{m}$-adic filtration of R and $I_1 := R_+ \neq R$.

If Δ is shellable, then R is a sequentially Cohen-Macaulay ring, so by Proposition 4.3 we get the following.

Proposition 5.1. *If Δ is shellable, then $\mathcal{R}'(\mathfrak{m})$ is a sequentially Cohen-Macaulay ring.*

Notice that $\mathfrak{p} \not\supseteq I_1$ for every $\mathfrak{p} \in \mathrm{Ass}R$ if and only if $F \neq \emptyset$ for all $F \in \mathcal{F}(\Delta)$, which is equivalent to saying that $\Delta \neq \{\emptyset\}$.

The goal of this section is the following. Here $|F_i|$ denotes the cardinality of F_i.

Theorem 5.2. *Suppose that Δ is shellable with shelling order $F_1, F_2, \ldots, F_m \in \mathcal{F}(\Delta)$ such that $\dim F_1 \ge \dim F_2 \ge \cdots \ge \dim F_m$ and $\Delta \neq \{\emptyset\}$. Then the following conditions are equivalent.*

(1) *$\mathcal{R}(\mathfrak{m})$ is a sequentially Cohen-Macaulay ring.*
(2) *$m = 1$ or if $m \ge 2$, then $|F_i| > \sharp\mathcal{F}(\Delta_1 \cap \Delta_2)$ for every $2 \le i \le m$, where $\Delta_1 = \langle F_1, F_2, \ldots, F_{i-1}\rangle$, $\Delta_2 = \langle F_i \rangle$.*

Proof. Thanks to Theorem 4.5, $\mathcal{R}$ is sequentially Cohen-Macaulay if and only if $\mathrm{a}(C_i) < 0$ for all $1 \le i \le \ell$, where $\{D_i\}_{0 \le i \le \ell}$ is the dimension filtration of R, $C_i = D_i/D_{i-1}$ and $d_i = \dim_R D_i$ for all $1 \le i \le \ell$. If $m = 1$, then $R = k[\Delta] \cong k[X_i \mid i \in F_1]$, which is a polynomial ring, so that $\ell = 1$ and $\mathrm{a}(R) = -|F_1| < 0$. Hence $\mathcal{R}$ is a Cohen-Macaulay ring by Lemma 4.4.

Suppose that $m > 1$ and the assertion holds for $m - 1$. We put $\Delta_1 = \langle F_1, F_2, \ldots, F_{m-1}\rangle$ and $\Delta_2 = \langle F_m \rangle$. If $\ell = 1$, then Δ is pure. Look at the following exact sequence

$$0 \to S/I_\Delta \to S/I_{\Delta_1} \oplus S/I_{\Delta_2} \to S/I_{\Delta_1} + I_{\Delta_2} \to 0$$

of graded R-modules. We then have

$$S/I_{\Delta_1} + I_{\Delta_2} \cong k[\Delta_2]/(\overline{\xi})$$

for some monomials $\xi \in I_{\Delta_1} \setminus I_{\Delta_2}$ in $X_1, X_2, \ldots, X_n$ with $0 < \deg \xi = \sharp\mathcal{F}(\Delta_1 \cap \Delta_2)$. Therefore $\mathrm{a}(S/I_{\Delta_1} + I_{\Delta_2}) = \sharp\mathcal{F}(\Delta_1 \cap \Delta_2) - |F_m|$. We put $\mathfrak{m} = R_+$. Then we have the exact sequence of local cohomology modules as follows

$$0 \to \mathrm{H}^{d-1}_{\mathfrak{m}}(S/I_{\Delta_1} + I_{\Delta_2}) \to \mathrm{H}^{d}_{\mathfrak{m}}(S/I_{\Delta}) \to \mathrm{H}^{d}_{\mathfrak{m}}(S/I_{\Delta_1}) \oplus \mathrm{H}^{d}_{\mathfrak{m}}(S/I_{\Delta_2}) \to 0.$$

Thus $\mathrm{a}(R) = \max\{\sharp\mathcal{F}(\Delta_1 \cap \Delta_2) - |F_m|, \mathrm{a}(k[\Delta_1]), \mathrm{a}(k[\Delta_2])\}$. Hence $\mathcal{R}$ is sequentially Cohen-Macaulay if and only if $\sharp\mathcal{F}(\Delta_1 \cap \Delta_2) < |F_m|$ and $\mathrm{a}(k[\Delta_1]) < 0$. By using the induction arguments, we get the equivalence between (1) and (2).

Suppose now that $\ell > 1$. Consider the following exact sequence

$$0 \to I_{\Delta_1}/I_{\Delta} \to S/I_{\Delta} \to S/I_{\Delta_1} \to 0$$

of graded R-modules. Then we have

$$I_{\Delta_1}/I_{\Delta} \cong I_{\Delta_1} + I_{\Delta_2}/I_{\Delta_2} = I_{\Delta_1 \cap \Delta_2}/I_{\Delta_2} = (\overline{\xi})$$

where $\xi \in I_{\Delta_1} \setminus I_{\Delta_2}$ is a homogeneous element with $0 < \deg \xi = \sharp\mathcal{F}(\Delta_1 \cap \Delta_2) =: t$. Therefore $I_{\Delta_1}/I_{\Delta} \cong S/I_{\Delta_2}(-t)$, so that

$$0 \to S/I_{\Delta_2}(-t) \xrightarrow{\sigma} S/I_{\Delta} \xrightarrow{\varepsilon} S/I_{\Delta_1} \to 0.$$

We put $L = \mathrm{Im}\sigma$. Then $L \neq (0)$, $\dim_R L = d_1$ and $\mathrm{a}(L) = t - |F_m|$. We notice here that $L \subseteq D_1$. Now we set $D_i{}' = \varepsilon(D_i)$ for every $1 \le i \le \ell$. Then $D_1{}' \subsetneq D_2{}' \subsetneq \ldots \subsetneq D_\ell{}' = k[\Delta_1]$ and $C_i{}' := D_i{}'/D_{i-1}{}' \cong C_i$ for all $2 \le i \le \ell$. Hence $\mathrm{a}(C_i) = \mathrm{a}(C_i{}')$ for $2 \le i \le \ell$.

<u>**Case 1**</u> $L \subsetneq D_1$ (i.e., $D_1{}' \neq (0)$)

In this case $D_0{}' := (0) \subsetneq D_1{}' \subsetneq D_2{}' \subsetneq \ldots \subsetneq D_\ell{}' = k[\Delta_1]$ is the dimension filtration of $k[\Delta_1]$. Look at the following exact sequence

$$0 \to L \to D_1 \to D_1' \to 0$$

of R-modules. Then $\mathrm{a}(D_1) = \max\{\mathrm{a}(L), \mathrm{a}(D_1{}')\}$.

<u>**Case 2**</u> $L = D_1$ (i.e., $D_1{}' = (0)$)

Similarly $(0) = D_1{}' \subsetneq D_2{}' \subsetneq \ldots \subsetneq D_\ell{}' = k[\Delta_1]$ is the dimension filtration of $k[\Delta_1]$.

Summing up, in any case $\mathcal{R}$ is a sequentially Cohen-Macaulay ring if and only if $\mathrm{a}(L) < 0$ and the assertion (1) holds for the ring $k[\Delta_1]$. Hence we get the equivalence of conditions (1) and (2) by using the induction hypothesis. □

Remark 5.3. If Δ is shellable, then we can take a shelling order $F_1, F_2, \ldots, F_m \in \mathcal{F}(\Delta)$ such that $\dim F_1 \ge \dim F_2 \ge \cdots \ge \dim F_m$.

Apply Theorem 5.2, we get the following.

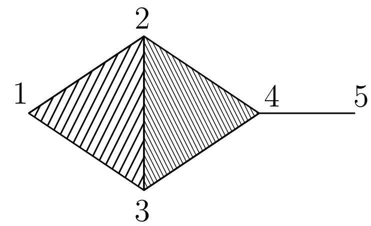
Corollary 5.4. *Under the same notation in Theorem 5.2. Suppose that* $|F_m| \ge 2$. *If* $\langle F_1, F_2, \ldots, F_{i-1}\rangle \cap \langle F_i\rangle$ *is a simplex for every* $2 \le i \le m$, *then* $\mathcal{R}(\mathfrak{m})$ *is a sequentially Cohen-Macaulay ring.*

Let us give some examples.

Example 5.5. Let $\Delta = \langle F_1, F_2, F_3\rangle$, where $F_1 = \{1, 2, 3\}$, $F_2 = \{2, 3, 4\}$ and $F_3 = \{4.5\}$. Then Δ is shellable with shelling order $F_1, F_2, F_3 \in \mathcal{F}(\Delta)$. Then

$$\langle F_1\rangle \cap \langle F_2\rangle\,, \quad \langle F_1, F_2\rangle \cap \langle F_3\rangle$$

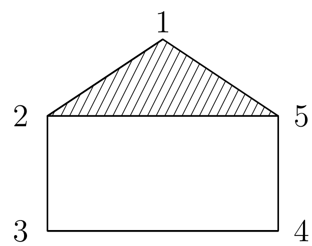
are simplexes, so that $\mathcal{R}(\mathfrak{m})$ is a sequentially Cohen-Macaulay ring.

$\Delta =$ 

Example 5.6. Let $\Delta = \langle F_1, F_2, F_3, F_4 \rangle$, where $F_1 = \{1, 2, 5\}$, $F_2 = \{2, 3\}$, $F_3 = \{3, 4\}$ and $F_4 = \{4, 5\}$. Notice that Δ is a shellable simplicial complex with shelling order $F_1, F_2, F_3, F_4 \in \mathcal{F}(\Delta)$. We put $\Delta_1 = \langle F_1, F_2, F_3 \rangle$ and $\Delta_2 = \langle F_4 \rangle$. Then

$$\sharp\mathcal{F}(\Delta_1 \cap \Delta_2) = 2 = |F_4|,$$

whence $\mathcal{R}(\mathfrak{m})$ is not sequentially Cohen-Macaulay.

$\Delta =$ 

Acknowledgments. *The authors would like to thank Professor Shiro Goto for his valuable advice and comments.*

References

[BR] J. W. Brewer and E. A. Rutter, *Must R be Noetherian if R^G is Noetherian*, Comm. Algebra, **5** (1977), 969-979.

[CC] N. T. Cuong and D. T. Cuong, *On sequentially Cohen-Macaulay modules,* Kodai Math. J., **(30)** (2007), 409-428.

[CGT] N. T. Cuong, S. Goto and H. L. Truong, *The equality $I^2 = \mathfrak{q}I$ in sequentially Cohen-Macaulay rings*, J. Algebra, **(379)** (2013), 50-79.

[F] G. Faltings, *Über die Annulatoren lokaler Kohomologiegruppen*, Archiv der Math., **30** (1978), 473–476.

[GHS] S. Goto, Y. Horiuchi and H. Sakurai, *Sequentially Cohen-Macaulayness versus parametric decomposition of powers of parameter ideals*, J. Comm. Algebra, **2** (2010), 37–54.

[GN] S. Goto and K. Nishida, *The Cohen-Macaulay and Gorenstein properties of Rees algebras associated to fltrations*, Mem. Amer. Math. Soc., **110** (1994).

[GS] S. Goto and Y. Shimoda, *On the Rees algebra of Cohen-Macaulay local rings*, Commutative Algebra, Lecture Note in Pure and Applied Mathematics, Marcel Dekker Inc, **68** (1982), 201-231.

[GW] S. Goto and K. Watanabe, *On graded rings, I*, J. Math. Soc. Japan, **30** (1978), 179–213.

[N] M. Nagata, Local Rings, Interscience, 1962.

[Sch] P. Schenzel, *On the dimension filtration and Cohen-Macaulay filtered modules*, in: Proc. of the Ferrara Meeting in honour of Mario Fiorentini, University of Antwerp, Wilrijk, Belgium, (1998) pp. 245-264.

[St] R. P. Stanley, Combinatorics and commutative algebra, Second Edition, Birkhäuser, Boston, 1996.

[TI] N. Trung and S. Ikeda, *When is the Rees algebra Cohen-Macaulay?*, Comm. Algebra **17** (1989), 2893-2922.

[V] D. Q. Viet, *A note on the Cohen-Macaulayness of Rees Algebra of filtrations*, Comm. Algebra **21** (1993), 221-229.

Department of Mathematics, School of Science and Technology, Meiji University, 1-1-1 Higashi-mita, Tama-ku, Kawasaki 214-8571, Japan
E-mail address: `taniguti@math.meiji.ac.jp`
URL: `http://www.isc.meiji.ac.jp/~taniguci/`

Department of Mathematics, School of Science and Technology, Meiji University, 1-1-1 Higashi-mita, Tama-ku, Kawasaki 214-8571, Japan
E-mail address: `sugarphuong@gmail.com`

Thai Nguyen University of Agriculture and Forestry, Thai Nguyen, Vietnam
E-mail address: `xsdung050764@gmail.com`

Thai Nguyen University of Pedagogical, Thai Nguyen, Vietnam
E-mail address: `antrannguyen@gmail.com`